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USING INVARIANTS TO EXTRACT GEOMETRIC CHARACTERISTICS OF CONIC SECTIONS FROM RATIONAL QUADRATIC PARAMETERIZATIONS

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ABSTRACT

Extracting the geometric characteristics of conic sections, such as their center, axes and foci, from their defining equations is required for various applications in computer graphics and geometric modeling. Although there exist standard techniques for computing the geometric characteristics for conics in implicit form, in shape modeling applications conic sections are often represented by rational quadratic parameterizations. Here we present closed formulas for computing the geometric characteristics of conics directly from their quadratic parameterizations without resorting to implicitization procedures. Our approach uses the invariants of rational quadratic parameterizations under rational linear reparameterizations. These invariants are also used to give a complete characterization of degenerate conics represented by rational quadratic parameterizations.

Keywords: Conic section; invariant; quadratic parameterization.

1. Introduction

Conic sections have many applications in computer graphics and computer aided design. In some applications it may be necessary to extract their geometric characteristics, such as their center, axes, and foci, from their defining equations. When conics are given in implicit form, it is well known how to compute these geometric characteristics, e.g. Ref. [4]. However, conic sections are frequently represented

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as rational quadratic parametric curves. It is therefore of practical importance to study how to compute their geometric characteristics directly from parametric representations, without first converting the conics into implicit form.

In Ref. [2], algorithms are derived for computing the center, foci, and axes of a conic represented in rational Bézier form. Lee's approach in Ref. [2] is highly geometric. Another geometric approach to determining the geometric characteristics of a conic section from its rational quadratic parameterization is presented by Albrecht in Ref. [1]. Albrecht's solution is based on some sophisticated results about conic sections from classical projective geometry. Both Lee and Albrecht take an algorithmic approach; neither provides simple closed form expressions for all the geometric characteristics of a conic section. In contrast, here we take an elementary algebraic approach, deriving our formulas from the invariants of rational quadratic parameterizations under rational linear reparameterizations. Our technique is conceptually straightforward and leads to simple explicit formulas for all the geometric characteristics of any conic section. These explicit formulas are presented in Theorems 1, 2 and 3. We also use invariants to give a complete characterization of degenerate conics (see Section 6). Therefore, our solution handles all rational quadratic parameterizations, including those giving rise to degenerate conics. Finally, our algebraic approach has the potential to extend to rational quadratic parameterizations of quadric surfaces, a topic we hope to address in a future paper.

2. Invariants and Classification

The geometric characteristics of a conic are independent of any particular parameterization of the conic. Since our goal is to express these characteristics in terms of (numerical) coefficients of a rational quadratic parameterization, it is natural to seek invariants of such a parameterization under rational linear reparameterization.

With homogeneous coordinates a conic section in the Euclidean plane is represented parametrically by

$$P(u,v) = u^2 E + 2uvF + v^2 G, \tag{1}$$

where $E = (e_x, e_y, e_w)$, $F = (f_x, f_y, f_w)$, $G = (g_x, g_y, g_w)$ and (u,v) are homogeneous parameters. We can rewrite this equation in matrix form

$$P(u,v) \equiv (p_x(u,v), p_y(u,v), p_w(u,v)) = (u,v)\,(N_x, N_y, N_w) \begin{pmatrix} u \\ v \end{pmatrix},$$

where

$$N_x = \begin{pmatrix} e_x & f_x \\ f_x & g_x \end{pmatrix}, \quad N_y = \begin{pmatrix} e_y & f_y \\ f_y & g_y \end{pmatrix}, \quad N_w = \begin{pmatrix} e_w & f_w \\ f_w & g_w \end{pmatrix}$$

are called the coefficient matrices of $P(u,v)$. Throughout this paper all coefficients are assumed to be real numbers.

Let $P(u,v)$ and $R(s,t)$ be two arbitrary rational quadratic parameterizations of the same conic. Denote the inverse of $P(u,v)$ by $\rho(u,v) = P^{-1}(x,y,w)$, where

(x, y, w) is a point on the conic. Then the reparameterization between $P(u, v)$ and $R(s, t)$ is given by $\rho(u, v) = P^{-1}(x, y, w) = P^{-1} \circ R(s, t)$. Clearly, $P^{-1} \circ R$ is a birational mapping from the real line into itself, thus is a projective transformation, i.e. a rational linear function (see Ref. [5]).

Let a general rational linear reparameterization be given by

$$(u, v) = (\tilde{u}, \tilde{v}) \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \equiv (\tilde{u}, \tilde{v})Q, \quad \text{with} \quad \det(Q) \neq 0. \tag{2}$$

Through this reparameterization we get another parameterization of the same conic,

$$\tilde{P}(\tilde{u}, \tilde{v}) = P(u(\tilde{u}, \tilde{v}), v((\tilde{u}, \tilde{v})) = (\tilde{u}, \tilde{v}) \left(QN_xQ^T, QN_yQ^T, QN_wQ^T\right) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}.$$

Denote the coefficient matrices of $\tilde{P}(\tilde{u}, \tilde{v})$ by $\tilde{N}_x = QN_xQ^T$, $\tilde{N}_y = QN_yQ^T$, and $\tilde{N}_w = QN_wQ^T$.

The invariants of $P(u, v)$ under the reparameterization (2) are, in fact, the invariants of the system of three quadratic binary forms determined by $P(u, v)$ under the linear projective transformations of the variables given by (2) (see Ref. [3]). To investigate these invariants, we shall need the notion of the adjoint of a 2×2 matrix. The adjoint of the matrix

$$W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is defined by

$$W^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Notice that $WW^* = \det(W)I$, where I is the identity matrix.

Lemma 1.

(a) $\det(\tilde{N}_\alpha) = \det^2(Q)\det(N_\alpha)$ *for any* $\alpha \in \{x, y, w\}$;

(b) If N_β *is nonsingular,* $\det(\tilde{N}_\alpha\tilde{N}_\beta^{-1}) = \det(N_\alpha N_\beta^{-1})$ *and* $\operatorname{tr}(\tilde{N}_\alpha\tilde{N}_\beta^{-1}) = \operatorname{tr}(N_\alpha N_\beta^{-1})$ *for any* $\alpha, \beta \in \{x, y, w\}$;

(c) $\operatorname{tr}(\tilde{N}_\alpha\tilde{N}_\beta^*) = \det^2(Q)\operatorname{tr}(N_\alpha N_\beta^*)$ *for any* $\alpha, \beta \in \{x, y, w\}$.

Proof. (a) This result follows from $\tilde{N}_\alpha = QN_\alpha Q^T$.

(b) This result follows because

$$\tilde{N}_\alpha\tilde{N}_\beta^{-1} = QN_\alpha Q^T(Q^T)^{-1}N_\beta^{-1}Q^{-1} = QN_\alpha N_\beta^{-1}Q^{-1},$$

so $\tilde{N}_\alpha\tilde{N}_\beta^{-1}$ and $N_\alpha N_\beta^{-1}$ have the same characteristic polynomial.

(c) Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to verify that $M^* = -JM^TJ$ and $JJ = -I$. So for any two 2×2 matrices A and B,

$$(AB)^* = -J(AB)^TJ = -JB^TA^TJ = (-JB^TJ)(-JA^TJ) = B^*A^*.$$

Since $\det(Q) \neq 0$,

$$\begin{aligned}\tilde{N}_\alpha \tilde{N}_\beta^* &= QN_\alpha Q^T(QN_\beta Q^T)^* = QN_\alpha Q^T(Q^T)^* N_\beta^* Q^* \\ &= \det{}^2(Q) QN_\alpha N_\beta^* Q^* / \det(Q) = \det{}^2(Q) QN_\alpha N_\beta^* Q^{-1}.\end{aligned}$$

Hence, $\mathrm{tr}(\tilde{N}_\alpha \tilde{N}_\beta^*) = \det^2(Q)\mathrm{tr}(N_\alpha N_\beta^*)$. □

By Lemma 1, $\det(N_\alpha)$ and $\mathrm{tr}(N_\alpha N_\beta^*)$ are invariants of weight 2, whereas $\det(N_\alpha N_\beta^{-1})$ and $\mathrm{tr}(N_\alpha N_\beta^{-1})$ are invariants of weight 0, for $\alpha, \beta \in \{x, y, w\}$ (see Ref. [3]). The geometric characteristics of a conic remain the same under a rational linear reparameterization, so they are also invariants. We are going to represent these geometric characteristics in terms of the invariants listed in Lemma 1.

There are three classes of non-degenerate conic sections: ellipses, parabolas, and hyperbolas. A conic in each class can be mapped affinely into any other member in the same class, but two conics in different classes cannot be mapped affinely into each other. A conic is an ellipse, parabola, or hyperbola if it intersects the line at infinity in no real point, a double real point, or two distinct real points, respectively. The number of real intersection points between a conic $P(u, v)$ in (1) and the line at infinity is determined by the number of real zeros of $p_w(u, v)$, since each real zero of $p_w(u, v)$ gives rise to a real point at infinity on the conic. Thus a conic can be classified by checking the sign of the discriminant of $p_w(u, v)$, which is $4f_w^2 - 4e_w g_w = -4\det(N_w)$. Hence, we obtain

Lemma 2. *A proper conic $P(u, v)$ in (1) is an ellipse, parabola, or hyperbola if* $\det(N_w) > 0$, $\det(N_w) = 0$, *or* $\det(N_w) < 0$, *respectively.*

Note that since $\det(\tilde{N}_w) = \det^2(Q)\det(N_w)$, the sign of $\det(N_w)$ is invariant under rational linear reparameterization.

3. Ellipses

By Lemma 2, an ellipse is signaled by $\det(N_w) > 0$. Let $P(u, v)$ in (1) be an arbitrary parameterization of the ellipse with center at $C = (c_x, c_y)$, semi-major axis vector $A = (a_x, a_y)$, and semi-minor axis vector $B = (b_x, b_y)$ – see Figure 1. Clearly, another parameterization of this ellipse is

$$\bar{P}(t) = \frac{1-t^2}{1+t^2}A + \frac{2t}{1+t^2}B + C,$$

since the ellipse can be generated by an affine transformation of the unit circle

$$(x(t), y(t)) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right).$$

With homogeneous parameters $(\tilde{u}, \tilde{v})$, where $\tilde{u}/\tilde{v} = t$, the corresponding homogeneous parameterization $\tilde{P}(\tilde{u}, \tilde{v}) = (\tilde{p}_x(\tilde{u}, \tilde{v}), \tilde{p}_y(\tilde{u}, \tilde{v}), \tilde{p}_w(\tilde{u}, \tilde{v}))$ is given by

$$\begin{aligned}
\tilde{p}_x(\tilde{u}, \tilde{v}) &= (\tilde{v}^2 - \tilde{u}^2)a_x + 2\tilde{u}\tilde{v}b_x + (\tilde{v}^2 + \tilde{u}^2)c_x \\
&= \tilde{u}^2(c_x - a_x) + 2\tilde{u}\tilde{v}b_x + \tilde{v}^2(c_x + a_x) = (\tilde{u}, \tilde{v})\tilde{N}_x \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \\
\tilde{p}_y(\tilde{u}, \tilde{v}) &= (\tilde{v}^2 - \tilde{u}^2)a_y + 2\tilde{u}\tilde{v}b_y + (\tilde{v}^2 + \tilde{u}^2)c_y = (\tilde{u}, \tilde{v})\tilde{N}_y \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \\
\tilde{p}_w(\tilde{u}, \tilde{v}) &= \tilde{v}^2 + \tilde{u}^2 = (\tilde{u}, \tilde{v})\tilde{N}_w \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix},
\end{aligned}$$

where

$$\tilde{N}_x = \begin{pmatrix} c_x - a_x & b_x \\ b_x & c_x + a_x \end{pmatrix}, \quad \tilde{N}_y = \begin{pmatrix} c_y - a_y & b_y \\ b_y & c_y + a_y \end{pmatrix}, \quad \tilde{N}_w = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The invariants expressed in the coefficients of $\tilde{P}(\tilde{u}, \tilde{v})$ are defined by the unknown vectors A, B, C, which in turn determine the sought after geometric characteristics. On the other hand, given an arbitrary quadratic parameterization $P(u, v)$ of the same conic as in (1), the invariants of $P(u, v)$ are expressions in the known coefficients of $P(u, v)$. By construction, these two sets of invariants are related. From these relations we will be able to solve for the geometric characteristics of the given conic in terms of the known coefficients of $P(u, v)$.

First, we solve for the unknown C in terms of the entries of N_x, N_y, and N_w. Consider

$$\tilde{N}_x \tilde{N}_w^{-1} = \begin{pmatrix} c_x - a_x & b_x \\ b_x & c_x + a_x \end{pmatrix}.$$

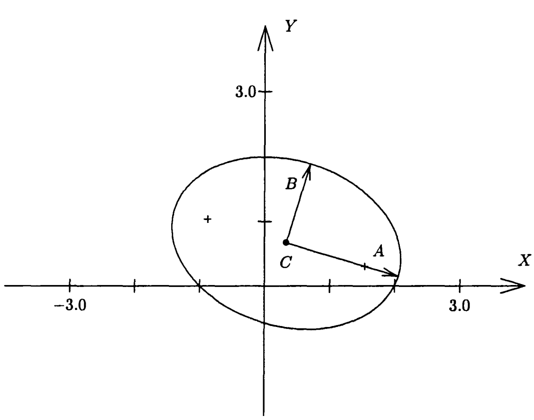

Fig. 1. The ellipse in Example 1. The center is marked by a bullet, and the foci by two crosses.

By Lemma 1(*b*),

$$\operatorname{tr}(N_x N_w^{-1}) = \operatorname{tr}(\tilde{N}_x \tilde{N}_w^{-1}) = 2c_x.$$

Hence,

$$c_x = \operatorname{tr}(N_x N_w^{-1})/2 = \frac{1}{2}\operatorname{tr}(N_x N_w^*)/\det(N_w).$$

Similarly,

$$c_y = \operatorname{tr}(N_y N_w^{-1})/2 = \frac{1}{2}\operatorname{tr}(N_y N_w^*)/\det(N_w).$$

To solve for the remaining unknowns A and B, consider

$$\det(\tilde{N}_x \tilde{N}_w^{-1}) = c_x^2 - a_x^2 - b_x^2.$$

By Lemma 1(*b*)

$$a_x^2 + b_x^2 = c_x^2 - \det(\tilde{N}_x \tilde{N}_w^{-1}) = c_x^2 - \det(N_x N_w^{-1})$$

Thus

$$a_x^2 + b_x^2 = \gamma_x, \tag{3}$$

where $\gamma_x = c_x^2 - \det(N_x N_w^{-1})$. Similarly,

$$a_y^2 + b_y^2 = \gamma_y, \tag{4}$$

where $\gamma_y = c_y^2 - \det(N_y N_w^{-1})$.

By Lemma 1(*a*), $\det(\tilde{N}_w) = \det^2(Q)\det(N_w)$. Since $\det(\tilde{N}_w) = 1$,

$$\det^2(Q) = \det(\tilde{N}_w)/\det(N_w) = 1/\det(N_w). \tag{5}$$

On the other hand, from

$$\tilde{N}_x \tilde{N}_y^* = \begin{pmatrix} c_x - a_x & b_x \\ b_x & c_x + a_x \end{pmatrix} \begin{pmatrix} c_y + a_y & -b_y \\ -b_y & c_y - a_y \end{pmatrix},$$

by Lemma 1(*c*) and (5), we obtain

$$2(c_x c_y - a_x a_y - b_x b_y) = \operatorname{tr}(\tilde{N}_x \tilde{N}_y^*) = \det^2(Q)\operatorname{tr}(N_x N_y^*) = \operatorname{tr}(N_x N_y^*)/\det(N_w).$$

Thus

$$a_x a_y + b_x b_y = \tau_{xy}, \tag{6}$$

where

$$\tau_{xy} = c_x c_y - \frac{1}{2}\operatorname{tr}(N_x N_y^*)/\det(N_w).$$

By (a) and (c) of Lemma 1, $\operatorname{tr}(N_x N_y^*)/\det(N_w) = \operatorname{tr}(\tilde{N}_x \tilde{N}_y^*)/\det(\tilde{N}_w)$ holds for any reparameterization Q. Hence, $\operatorname{tr}(N_x N_y^*)/\det(N_w) = \operatorname{tr}(N_x N_y^{-1})/\det(N_w N_y^{-1})$ is an invariant, assuming N_y is invertible.

To solve equations (3), (4), and (6), let $\ell_1 = |A|$ and $\ell_2 = |B|$. In addition, let θ denote the angle between the semi-major axis vector A and the x-axis, where $\theta \in (-\pi/2, \pi/2]$, so that (ℓ_1, θ) are the polar coordinates of A. Then $a_x = \ell_1 \cos\theta$, $a_y = \ell_1 \sin\theta$, $b_x = \ell_2 \cos(\theta + \pi/2) = -\ell_2 \sin\theta$, and $b_y = \ell_2 \sin(\theta + \pi/2) = \ell_2 \cos\theta$. Substituting these values into equations (3), (4), and (6), we obtain

$$\ell_1^2 \cos^2\theta + \ell_2^2 \sin^2\theta = \gamma_x, \tag{7}$$

$$\ell_1^2 \sin^2\theta + \ell_2^2 \cos^2\theta = \gamma_y, \tag{8}$$

$$\ell_1^2 \sin\theta\cos\theta - \ell_2^2 \sin\theta\cos\theta = \tau_{xy}. \tag{9}$$

We are now going to solve equations (7), (8) and (9). Subtracting (8) from (7) yields

$$(\ell_1^2 - \ell_2^2)\cos 2\theta = \gamma_x - \gamma_y. \tag{10}$$

Moreover, it follows from (9) that

$$(\ell_1^2 - \ell_2^2)\sin 2\theta = 2\tau_{xy}. \tag{11}$$

When $\ell_1 = \ell_2$, the ellipse is a circle, and the angle θ is undefined. Suppose however that $\ell_1 > \ell_2 > 0$. Squaring and adding equations (10) and (11) leads to

$$\ell_1^2 - \ell_2^2 = \sqrt{(\gamma_x - \gamma_y)^2 + 4\tau_{xy}^2} \equiv \delta_{xy}, \tag{12}$$

Adding (7) to (8) yields

$$\ell_1^2 + \ell_2^2 = \gamma_x + \gamma_y. \tag{13}$$

Finally, adding and subtracting (13) and (12) yields

$$\ell_1^2 = \frac{1}{2}\left[\gamma_x + \gamma_y + \sqrt{(\gamma_x - \gamma_y)^2 + 4\tau_{xy}^2}\right] = \frac{1}{2}(\gamma_x + \gamma_y + \delta_{xy})$$

and

$$\ell_2^2 = \frac{1}{2}\left[\gamma_x + \gamma_y - \sqrt{(\gamma_x - \gamma_y)^2 + 4\tau_{xy}^2}\right] = \frac{1}{2}(\gamma_x + \gamma_y - \delta_{xy}).$$

Since by equation (12)

$$\ell_1^2 - \ell_2^2 = \delta_{xy},$$

it follows by equations (10) and (11) that

$$\cos 2\theta = \frac{\gamma_x - \gamma_y}{\delta_{xy}} \quad \text{and} \quad \sin 2\theta = \frac{2\tau_{xy}}{\delta_{xy}}.$$

By (10) and (11), the ellipse is a circle if and only if $\gamma_x - \gamma_y = \tau_{xy} = 0$. In this case, it follows from equations (7) and (8) that the radius of the circle is given by

$$\begin{aligned} r = \ell_1 = \sqrt{\gamma_x} &= [c_x^2 - \det(N_x)/\det(N_w)]^{1/2} \\ = \sqrt{\gamma_y} &= [c_y^2 - \det(N_y)/\det(N_w)]^{1/2}. \end{aligned}$$

The above results are summarized in the theorem below.

Theorem 1. *Let $\mathcal{E}$ denote an ellipse given by $P(u,v)$ in (1), i.e.* $\det(N_w) > 0$.

(a) The center $C = (c_x, c_y)$ of $\mathcal{E}$ is given by

$$c_x = \frac{1}{2}\mathrm{tr}(N_x N_w^*)/\det(N_w),$$
$$c_y = \frac{1}{2}\mathrm{tr}(N_y N_w^*)/\det(N_w).$$

(b) If $\mathcal{E}$ is not a circle, the semi-major axis vector $A = (a_x, a_y)$ and semi-minor axis vector $B = (b_x, b_y)$ of $\mathcal{E}$ are given by

$$\begin{aligned} a_x &= \ell_1 \cos\theta, \\ a_y &= \ell_1 \sin\theta, \\ b_x &= -\ell_2 \sin\theta, \\ b_y &= \ell_2 \cos\theta, \end{aligned}$$

where

$$\ell_1 = \frac{1}{\sqrt{2}}[\gamma_x + \gamma_y + \delta_{xy}]^{1/2},$$
$$\ell_2 = \frac{1}{\sqrt{2}}[\gamma_x + \gamma_y - \delta_{xy}]^{1/2},$$
$$\cos 2\theta = (\gamma_x - \gamma_y)/\delta_{xy} \quad \text{and} \quad \sin 2\theta = 2\tau_{xy}/\delta_{xy}.$$

Here

$$\begin{aligned} \gamma_x &= c_x^2 - \det(N_x)/\det(N_w), \\ \gamma_y &= c_y^2 - \det(N_y)/\det(N_w), \\ \tau_{xy} &= c_x c_y - \frac{1}{2}\mathrm{tr}(N_x N_y^*)/\det(N_w), \\ \delta_{xy} &= \sqrt{(\gamma_x - \gamma_y)^2 + 4\tau_{xy}^2}. \end{aligned}$$

(c) The eccentricity of $\mathcal{E}$ is

$$e = \sqrt{1 - \ell_2^2/\ell_1^2} = \sqrt{\frac{2\delta_{xy}}{\gamma_x + \gamma_y + \delta_{xy}}},$$

and the foci of $\mathcal{E}$ are at

$$(c_x, c_y) \pm e(a_x, a_y).$$

(d) $\mathcal{E}$ is a circle if and only if $\delta_{xy} = 0$, or equivalently, $\gamma_x - \gamma_y = \tau_{xy} = 0$. When $\mathcal{E}$ is a circle, its radius is

$$r = \sqrt{\gamma_x} = \sqrt{\gamma_y}.$$

If $\gamma_x = \gamma_y = 0$, *then* $r = 0$ *and* $\mathcal{E}$ *collapses to a single point* (c_x, c_y).

Remark: Once $\cos 2\theta$ and $\sin 2\theta$ are known, one may find $\cos\theta$ and $\sin\theta$, where $\theta \in (-\pi/2, \pi/2]$, by the half angle formulas:

$$\cos\theta = \sqrt{(1+\cos 2\theta)/2},$$
$$\sin\theta = \begin{cases} \sqrt{(1-\cos 2\theta)/2} & \text{if } \sin 2\theta \geq 0; \\ -\sqrt{(1-\cos 2\theta)/2} & \text{if } \sin 2\theta < 0. \end{cases}$$

Example 1. Consider the rational quadratic curve

$$P(t) = (p_x(t),\ p_y(t),\ p_w(t)) = (-t^2+3t,\ -2t+2,\ t^2-t+1),$$

which is shown in Figure 1. Here

$$N_x = \begin{pmatrix} -1 & 3/2 \\ 3/2 & 0 \end{pmatrix},\ N_y = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix},\ N_w = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix};$$

therefore

$$\det(N_x) = -9/4,\ \det(N_y) = -1,\ \det(N_w) = 3/4,$$
$$\operatorname{tr}(N_x N_w^*) = 1/2,\ \operatorname{tr}(N_y N_w^*) = 1,\ \operatorname{tr}(N_x N_y^*) = 1.$$

Since $\det(N_w) = 3/4 > 0$, by Lemma 2 the curve is an ellipse. The center of this ellipse is at

$$C = (c_x, c_y) = \left(\frac{1}{2}\operatorname{tr}(N_x N_w^*)/\det(N_w),\ \frac{1}{2}\operatorname{tr}(N_y N_w^*)/\det(N_w)\right) = (1/3, 2/3).$$

Moreover,

$$\gamma_x = c_x^2 - \det(N_x)/\det(N_w) = 28/9,$$
$$\gamma_y = c_y^2 - \det(N_y)/\det(N_w) = 16/9,$$
$$\tau_{xy} = c_x c_y - \frac{1}{2}\operatorname{tr}(N_x N_y^*)/\det(N_w) = -4/9,$$
$$\delta_{xy} = \sqrt{(\gamma_x - \gamma_y)^2 + 4\tau_{xy}^2} = 4\sqrt{13}/9.$$

Thus

$$\ell_1 = \frac{1}{3}[22+\sqrt{52}]^{1/2} \approx 1.801577, \qquad \ell_2 = \frac{1}{3}[22-\sqrt{52}]^{1/2} \approx 1.281878\ .$$

To find θ, we first have

$$\cos 2\theta = (\gamma_x - \gamma_y)/\delta_{xy} = \frac{3}{\sqrt{13}},$$
$$\sin 2\theta = 2\tau_{xy}/\delta_{xy} = -\frac{2}{\sqrt{13}}.$$

Hence,

$$\cos\theta \approx 0.957092, \qquad \sin\theta \approx -0.289784$$

So the semi-major axis vector is

$$A = (a_x, a_y) = (\ell_1 \cos\theta,\ \ \ell_1 \sin\theta) \approx (1.724275,\ \ -0.522068),$$

and the semi-minor axis vector is

$$B = (b_x, b_y) = (-\ell_2 \sin\theta,\ \ \ell_2 \cos\theta) \approx (0.371468,\ \ 1.226875).$$

The eccentricity is

$$e = \sqrt{1 - \ell_2^2/\ell_1^2} = 0.702655$$

and the two foci are located at (1.544903, 0.299833) and (−0.878236, 1.033500).

4. Hyperbolas

By Lemma 2, a hyperbola is signaled by $\det(N_w) < 0$. Let $P(u,v)$ in (1) be an arbitrary parameterization of the hyperbola with center at $C = (c_x, c_y)$, semi-major axis vector $A = (a_x, a_y)$, and semi-minor axis vector $B = (b_x, b_y)$ (see Figure 2). Another parameterization of the hyperbola is

$$\bar{P}(t) = \frac{1+t^2}{1-t^2}A + \frac{2t}{1-t^2}B + C.$$

By the same derivation that we used for the ellipse, we can also obtain the following formulas for the features of the hyperbola. The details of this derivation are omitted, since they are much the same as the details for the ellipse.

Theorem 2. *Let $\mathcal{H}$ denote a hyperbola given by $P(u,v)$ in (1), i.e.* $\det(N_w) < 0$.

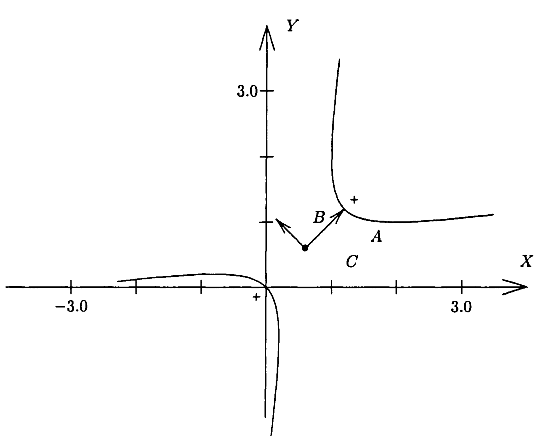

Fig. 2. The hyperbola in Example 2. The center is marked by a bullet, and the foci by two crosses.

(a) The center $C = (c_x, c_y)$ *of* $\mathcal{H}$ *is given by*

$$c_x = \frac{1}{2}\mathrm{tr}(N_x N_w^*)/\det(N_w),$$
$$c_y = \frac{1}{2}\mathrm{tr}(N_y N_w^*)/\det(N_w).$$

(b) The semi-major axis vector $A = (a_x, a_y)$ *and semi-minor axis vector* $B = (b_x, b_y)$ *of* $\mathcal{H}$ *are given by*

$$a_x = \ell_1 \cos\theta,$$
$$a_y = \ell_1 \sin\theta,$$
$$b_x = -\ell_2 \sin\theta,$$
$$b_y = \ell_2 \cos\theta,$$

where

$$\ell_1 = \frac{1}{\sqrt{2}}[\gamma_x + \gamma_y + \delta_{xy}]^{1/2}, \tag{14}$$

$$\ell_2 = \frac{1}{\sqrt{2}}[-(\gamma_x + \gamma_y) + \delta_{xy}]^{1/2}, \tag{15}$$

$$\cos 2\theta = (\gamma_x - \gamma_y)/\delta_{xy} \quad \text{and} \quad \sin 2\theta = 2\tau_{xy}/\delta_{xy}. \tag{16}$$

Here

$$\gamma_x = c_x^2 - \det(N_x)/\det(N_w),$$
$$\gamma_y = c_y^2 - \det(N_y)/\det(N_w),$$
$$\tau_{xy} = c_x c_y - \frac{1}{2}\mathrm{tr}(N_x N_y^*)/\det(N_w),$$
$$\delta_{xy} = \sqrt{(\gamma_x - \gamma_y)^2 + 4\tau_{xy}^2}.$$

(c) The eccentricity of $\mathcal{H}$ *is*

$$e = \sqrt{1 + \ell_2^2/\ell_1^2} = \sqrt{\frac{2\delta_{xy}}{\gamma_x + \gamma_y + \delta_{xy}}},$$

and the foci of $\mathcal{H}$ *are at*

$$(c_x, c_y) \pm e(a_x, a_y).$$

(d) The two asymptotes of $\mathcal{H}$ *pass through the center* (c_x, c_y) *and have direction vectors*

$$D_{1,2} = A \pm B = (\ell_1 \cos\theta \mp \ell_2 \sin\theta, \ell_1 \sin\theta \pm \ell_2 \cos\theta).$$

Remark: It will be shown in Section 6 (Theorem 7) that $\delta_{xy} \neq 0$ for a proper hyperbola. Thus the expressions for $\sin 2\theta$ and $\cos 2\theta$ in equation (16) are always well-defined for a proper hyperbola.

Example 2. Consider the rational quadratic curve

$$P(t) = (p_x(t),\ p_y(t),\ p_w(t)) = (-t + 2,\ t + 1,\ -t^2 + t + 1),$$

which is shown in Figure 2. Here

$$N_x = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 2 \end{pmatrix}, \quad N_y = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \quad N_w = \begin{pmatrix} -1 & 1/2 \\ 1/2 & 1 \end{pmatrix};$$

therefore

$$\det(N_x) = -1/4, \ \det(N_y) = -1/4, \ \det(N_w) = -5/4,$$
$$\operatorname{tr}(N_x N_w^*) = -3/2, \ \operatorname{tr}(N_y N_w^*) = -3/2, \ \operatorname{tr}(N_x N_y^*) = 1/2.$$

Since $\det(N_w) = -5/4 < 0$, by Lemma 2 the curve is a hyperbola. The center of this hyperbola is at

$$C = (c_x, c_y) = \left(\frac{1}{2}\operatorname{tr}(N_x N_w^*)/\det(N_w), \ \frac{1}{2}\operatorname{tr}(N_y N_w^*)/\det(N_w)\right) = (3/5, 3/5).$$

Moreover,

$$\begin{aligned} \gamma_x &= c_x^2 - \det(N_x)/\det(N_w) = 4/25, \\ \gamma_y &= c_y^2 - \det(N_y)/\det(N_w) = 4/25, \\ \tau_{xy} &= c_x c_y - \frac{1}{2}\operatorname{tr}(N_x N_y^*)/\det(N_w) = 14/25, \\ \delta_{xy} &= \sqrt{(\gamma_x - \gamma_y)^2 + 4\tau_{xy}^2} = 28/25. \end{aligned}$$

Thus

$$\ell_1 = \frac{3\sqrt{2}}{5}, \qquad \ell_2 = \frac{\sqrt{10}}{5} \ .$$

To find θ, we first have

$$\begin{aligned} \cos 2\theta &= (\gamma_x - \gamma_y)/\delta_{xy} = 0, \\ \sin 2\theta &= 2\tau_{xy}/\delta_{xy} = 1 \ . \end{aligned}$$

Hence,

$$\cos\theta = \frac{\sqrt{2}}{2}, \qquad \sin\theta = \frac{\sqrt{2}}{2}.$$

The two symmetry axis vectors are

$$\begin{aligned} A &= (a_x, a_y) = (\ell_1 \cos\theta, \ \ell_1 \sin\theta) = (0.6, \ 0.6), \\ B &= (b_x, b_y) = (-\ell_2 \sin\theta, \ \ell_2 \cos\theta) \approx (-0.447214, \ 0.447214). \end{aligned}$$

The directions of the two asymptotes are

$$\begin{aligned} D_1 &= A + B \approx (0.152786, \ 1.047214), \\ D_2 &= A - B \approx (1.047214, \ 0.152786). \end{aligned}$$

The eccentricity is

$$e = \sqrt{1 + \ell_2^2/\ell_1^2} = \frac{\sqrt{14}}{3} \approx 1.247219$$

and the two foci are located at (1.348331, 1.348331) and (−0.148331, −0.148331).

5. Parabolas

By Lemma 2, a parabola is signaled by $\det(N_w) = 0$. Any parabola can be mapped by a Euclidean transformation into a standard parabola $y^2 = 4kx$, whose focus is at $(k, 0)$, for some $k > 0$ (see Ref. [4]). Let $\lambda = \sqrt{k}$. Then one parameterization of the parabola $y^2 = 4kx$ is

$$R(t) = (1,0)t^2 + 2\lambda(0,1)t.$$

Let $P(u, v)$ in (1) be an arbitrary parameterization of a parabola $\mathcal{P}$ with the direction vector of its symmetry axis $D = (d_x, d_y)$, where $||D|| = 1$, and vertex $V = (v_x, v_y)$ – see Figure 3. Then, by the preceding discussion, a standard parameterization of $\mathcal{P}$ is

$$\bar{P}(t) = Dt^2 + 2\lambda Ht + V,$$

where $H \equiv (h_x, h_y) = (-d_y, d_x)$. The corresponding homogeneous form is

$$\tilde{P}(\tilde{u}, \tilde{v}) = D\tilde{u}^2 + 2\lambda H\tilde{u}\tilde{v} + V\tilde{v}^2,$$

so here

$$\tilde{N}_x = \begin{pmatrix} d_x & \lambda h_x \\ \lambda h_x & v_x \end{pmatrix}, \quad \tilde{N}_y = \begin{pmatrix} d_y & \lambda h_y \\ \lambda h_y & v_y \end{pmatrix}, \quad \tilde{N}_w = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Our goal is to solve for $(d_x, d_y, 0)$, $(h_x, h_y, 0)$, $(v_x, v_y, 1)$, and k in terms of the entries of N_x, N_y, N_w. Consider

$$\tilde{N}_x \tilde{N}_w^* = \begin{pmatrix} d_x & \lambda h_x \\ \lambda h_x & v_x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} d_x & 0 \\ \lambda h_x & 0 \end{pmatrix}.$$

By Lemma 1(c),

$$d_x = \operatorname{tr}(\tilde{N}_x \tilde{N}_w^*) = \det^2(Q)\operatorname{tr}(N_x N_w^*).$$

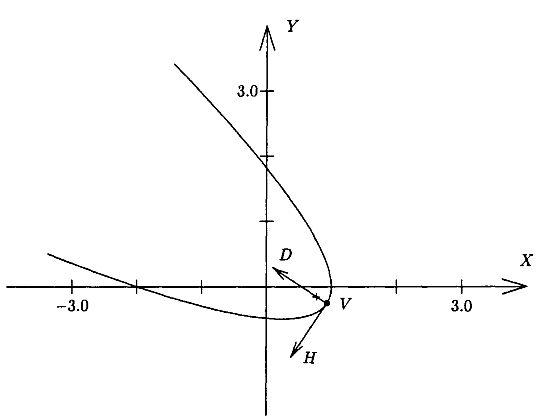

Fig. 3. The parabola in Example 3. The vertex is marked by a bullet, and the focus by a cross.

Similarly

$$d_y = \det^2(Q)\mathrm{tr}(N_y N_w^*).$$

Since $d_x^2 + d_y^2 = 1$,

$$\det^2(Q) = \frac{1}{[\mathrm{tr}^2(N_x N_w^*) + \mathrm{tr}^2(N_y N_w^*)]^{1/2}} = \frac{1}{L}, \tag{17}$$

where $L = [\mathrm{tr}^2(N_x N_w^*) + \mathrm{tr}^2(N_y N_w^*)]^{1/2}$. Thus

$$D = (d_x, d_y) = (\mathrm{tr}(N_x N_w^*)/L,\ \ \mathrm{tr}(N_y N_w^*)/L),$$

and

$$H = (h_x, h_y) = (-\mathrm{tr}(N_y N_w^*)/L,\ \ \mathrm{tr}(N_x N_w^*)/L).$$

By Lemma 1(a) and equation (17),

$$d_x v_x - h_x^2 \lambda^2 = \det(\tilde{N}_x) = \det^2(Q)\det(N_x) = \det(N_x)/L. \tag{18}$$

Similarly,

$$d_y v_y - h_y^2 \lambda^2 = \det(N_y)/L. \tag{19}$$

Furthermore, by Lemma 1(c) and equation (17),

$$d_x v_y + d_y v_x - 2h_x h_y \lambda^2 = \mathrm{tr}(\tilde{N}_x \tilde{N}_y^*) = \det^2(Q)\mathrm{tr}(N_x N_y^*) = \mathrm{tr}(N_x N_y^*)/L. \tag{20}$$

Let $c = d_x = h_y$ and $s = d_y = -h_x$. Then the three equations (18), (19), and (20) for λ, v_x, and v_y can be rewritten as

$$cv_x - s^2\lambda^2 = \gamma_x, \tag{21}$$

$$sv_y - c^2\lambda^2 = \gamma_y, \tag{22}$$

$$sv_x + cv_y + 2cs\lambda^2 = \tau_{xy}, \tag{23}$$

where

$$\gamma_x = \det(N_x)/L,\ \ \gamma_y = \det(N_y)/L,\ \ \tau_{xy} = \mathrm{tr}(N_x N_y^*)/L.$$

Now we are going to solve equations (21), (22) and (23) for λ, v_x, and v_y. Multiplying both sides of (23) by cs yields

$$cs^2 v_x + c^2 s v_y + 2c^2 s^2 \lambda^2 = cs\tau_{xy}, \tag{24}$$

Substituting $cv_x = \gamma_x + s^2\lambda^2$ and $sv_y = \gamma_y + c^2\lambda^2$ into (24), we obtain

$$cs\tau_{xy} = s^2(\gamma_x + s^2\lambda^2) + c^2(\gamma_y + c^2\lambda^2) + 2c^2s^2\lambda^2 = s^2\gamma_x + c^2\gamma_y + \lambda^2,$$

since $c^2 + s^2 = d_x^2 + d_y^2 = 1$. Hence,

$$k = \lambda^2 = cs\tau_{xy} - s^2\gamma_x - c^2\gamma_y.$$

From (21) and (22) it follows that

$$v_x = \frac{1}{c}[\gamma_x + s^2(cs\tau_{xy} - s^2\gamma_x - c^2\gamma_y)] = s^3\tau_{xy} + c(1 + s^2)\gamma_x - cs^2\gamma_y,$$

$$v_y = \frac{1}{s}[\gamma_y + c^2(cs\tau_{xy} - s^2\gamma_x - c^2\gamma_y)] = c^3\tau_{xy} - c^2 s\gamma_x + s(1+c^2)\gamma_y.$$

Furthermore, according to the discussion at the beginning of this section, the focus (r_x, r_y) of $\mathcal{P}$ is $V + kD$. Hence,

$$\begin{aligned} r_x = v_x + kd_x &= s^3\tau_{xy} + c(1+s^2)\gamma_x - cs^2\gamma_y + c(cs\tau_{xy} - s^2\gamma_x - c^2\gamma_y) \\ &= s\tau_{xy} + c(\gamma_x - \gamma_y). \end{aligned}$$

Similarly,

$$r_y = v_y + kd_y = c\tau_{xy} - s(\gamma_x - \gamma_y).$$

These results are summarized in the following theorem.

Theorem 3. *Let $\mathcal{P}$ denote a parabola given by $P(u,v)$ in (1), i.e.* $\det(N_w) = 0$.

(a) The symmetry axis of $\mathcal{P}$ has the unit direction vector $D = (c, s)$, where $c = \mathrm{tr}(N_x N_w^*)/L$, $s = \mathrm{tr}(N_y N_w^*)/L$, *and* $L = [\mathrm{tr}^2(N_x N_w^*) + \mathrm{tr}^2(N_y N_w^*)]^{1/2}$.

(b) The vertex $V = (v_x, v_y)$ of $\mathcal{P}$ is given by

$$\begin{aligned} v_x &= s^3\tau_{xy} + c(1+s^2)\gamma_x - cs^2\gamma_y, \\ v_y &= c^3\tau_{xy} - c^2 s\gamma_x + s(1+c^2)\gamma_y, \end{aligned}$$

where

$$\gamma_x = \det(N_x)/L, \quad \gamma_y = \det(N_y)/L, \quad \tau_{xy} = \mathrm{tr}(N_x N_y^*)/L.$$

(c) The focus (r_x, r_y) of $\mathcal{P}$ is given by

$$\begin{aligned} r_x &= s\tau_{xy} + c(\gamma_x - \gamma_y), \\ r_y &= c\tau_{xy} - s(\gamma_x - \gamma_y). \end{aligned}$$

(d) The focal length k of $\mathcal{P}$ is given by

$$k = cs\tau_{xy} - s^2\gamma_x - c^2\gamma_y.$$

Remark: We shall see in Section 6 (Lemma 7) that for a proper parabola $L \neq 0$ always holds. Thus for a proper parabola the expressions for c, s, γ_x, γ_y, and τ_{xy} in Theorem 3 are always well-defined.

Example 3. Consider the quadratic polynomial curve

$$P(t) = (p_x(t),\ p_y(t),\ p_w(t)) = (-3t^2 + 6t - 2,\ 2t^2 - 2t,\ 1),$$

which is shown in Figure 3. Here

$$N_x = \begin{pmatrix} -3 & 3 \\ 3 & -2 \end{pmatrix}, \quad N_y = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}, \quad N_w = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\det(N_x) = -3, \quad \det(N_y) = -1, \quad \det(N_w) = 0,$$

$$\mathrm{tr}(N_x N_w^*) = -3, \quad \mathrm{tr}(N_y N_w^*) = 2, \quad \mathrm{tr}(N_x N_y^*) = 2.$$

Since $\det(N_w) = 0$, by Lemma 2 the curve is a parabola. Now

$$L = [\mathrm{tr}^2(N_x N_w^*) + \mathrm{tr}^2(N_y N_w^*)]^{1/2} = \sqrt{13}.$$

$$c = \mathrm{tr}(N_x N_w^*)/L = -\frac{3}{\sqrt{13}}, \quad s = \mathrm{tr}(N_y N_w^*)/L = \frac{2}{\sqrt{13}}.$$

$$\gamma_x = \det(N_x)/L = -3/\sqrt{13}, \quad \gamma_y = \det(N_y)/L = -1/\sqrt{13},$$
$$\tau_{xy} = \mathrm{tr}(N_x N_y^*)/L = 2/\sqrt{13}.$$

Thus the direction of the symmetry axis vector is

$$D = (c, s) = (-3/\sqrt{13}, 2/\sqrt{13}),$$

and

$$H = (-s, c) = (-2/\sqrt{13}, -3/\sqrt{13}).$$

Moreover, the vertex is located at

$$(v_x, v_y) \approx (0.928994, \ -0.260355),$$

and the focus is at

$$(r_x, r_y) \approx (0.769231, \ -0.153846).$$

6. Degenerate Cases

When the conic $P(u,v)$ given in (1) is degenerate, the computations in Theorems 1-3 for extracting the geometric characteristics of a conic section break down. Therefore it is vital to detect any degeneracy in $P(u,v)$. Both the detection as well as a complete characterization of these degenerate cases are discussed in this section.

A conic in the form of Equation (1) can become degenerate in one of three ways:

(1) The three components $p_x(u,v)$, $p_y(u,v)$, $p_w(u,v)$ of $P(u,v)$ have exactly one common linear factor. In this case, $P(u,v)$ is a rational linear curve, i.e. $P(u,v)$ represents a line. One example of such a curve is

$$p_x(u,v) = u(u-v)$$
$$p_y(u,v) = u(u+v)$$
$$p_w(u,v) = 2uv,$$

which represents the line $x - y + w = 0$.

(2) The three components $p_x(u,v)$, $p_y(u,v)$ and $p_w(u,v)$ of $P(u,v)$ share a common factor of degree two – that is, $p_x(u,v)$, $p_y(u,v)$ and $p_w(u,v)$ are scalar multiples of each other. In this case, $P(u,v)$ is a rational curve of degree zero, i.e. $P(u,v)$ represents a single point. One example of such a curve is

$$p_x(u,v) = 2u(u-v)$$
$$p_y(u,v) = 2u(u-v)$$
$$p_w(u,v) = u(u-v),$$

which represents the point with homogeneous coordinates $(2, 2, 1)$.

(3) $P(u, v)$ is not a faithful parameterization – that is, $P(u, v)$ is 2-to-1 almost everywhere; such a parameterization $P(u, v)$ is also said to be *unfaithful.* An example is

$$\begin{aligned} p_x(u, v) &= u^2 \\ p_y(u, v) &= v^2 \\ p_w(u, v) &= u^2 + v^2, \end{aligned}$$

which represents the line $x + y - w = 0$. We assume that the three components of an unfaithful parameterization $P(u, v)$ in the form of Equation (1) do not have a common factor; for if they do, by the preceding discussion, $P(u, v)$ would represent either a line (a 1-to-1 mapping after the common factor is removed) or a single point (an ∞-to-1 mapping).

To study these degenerate cases, we introduce the matrix

$$M = \begin{pmatrix} e_x & e_y & e_w \\ 2f_x & 2f_y & 2f_w \\ g_x & g_y & g_w \end{pmatrix}.$$

Notice that

$$P(u, v) = (u^2, uv, v^2)M. \tag{25}$$

Theorem 4. *A rational quadratic parameterization $P(u, v)$ in the form of Equation (1) is degenerate if and only if* $\det(M) = 0$. *A degenerate parameterization $P(u, v)$ represents either a line or a single point. Moreover, $P(u, v)$ represents a line if and only if rank$(M) = 2$, and $P(u, v)$ represents a single point if and only if rank$(M) = 1$.*

Proof. Suppose that $P(u, v)$ is degenerate. According to the preceding discussion, $P(u, v)$ represents either a line or a single point. Therefore there exists a linear equation $ax + by + cw = 0$ that is satisfied by all the points on $P(u, v)$, i.e.

$$ap_x(u, v) + bp_y(u, v) + cp_w(u, v) = 0,$$

where a, b, and c are not all zero. Therefore

$$(u^2, uv, v^2)M(a, b, c)^T = 0$$

for all (u, v). It follows that $M(a, b, c)^T = 0$. Hence, $\det(M) = 0$.

Conversely, suppose that $\det(M) = 0$. Then there exist $(a, b, c) \neq 0$ such that $M(a, b, c)^T = 0$. Therefore

$$(u^2, uv, v^2)M(a, b, c)^T = 0$$

for all (u, v). Thus $ap_x(u, v) + bp_y(u, v) + cp_w(u, v) = 0$. That is, $P(u, v)$ is contained in the line $ax + by + cw = 0$. Hence, $P(u, v)$ is degenerate.

From the three cases of degeneracy listed at the beginning of this section, we see that a degenerate parameterization $P(u,v)$ represents either a line or a single point. Clearly, $rank(M) = 1$ if and only if the three components of $P(u,v)$ are scalar multiples of each other, i.e. if and only if $P(u,v)$ represents a single point.

Thus $rank(M) = 2$ if and only if $P(u,v)$ represents a line – that is, if and only if $P(u,v)$ is either an unfaithful parameterization or the three components of $P(u,v)$ have exactly one common linear factor. □

Remark: The rank of M is invariant under reparameterization. Indeed, under reparameterization (2),

$$(u^2, uv, v^2) = (\tilde{u}^2, \tilde{u}\tilde{v}, \tilde{v}^2)R,$$

where

$$R = \begin{pmatrix} \alpha^2 & \alpha\gamma & \gamma^2 \\ 2\alpha\beta & \alpha\delta + \beta\gamma & 2\gamma\delta \\ \beta^2 & \beta\delta & \delta^2 \end{pmatrix}.$$

It is easy to verify that $\det(R) = \det^3(Q) \neq 0$. Since

$$P(u,v) = (u^2, uv, v^2)M = (\tilde{u}^2, \tilde{u}\tilde{v}, \tilde{v}^2)RM = (\tilde{u}^2, \tilde{u}\tilde{v}, \tilde{v}^2)\tilde{M} = \tilde{P}(\tilde{u}, \tilde{v}),$$

we have $\tilde{M} = RM$. Hence, $rank(M)$ is invariant under reparameterization.

Theorem 4 provides a way to detect if a rational quadratic parameterization is degenerate. Only in the case where $\det(M) \neq 0$, should one continue to apply Theorem 1, 2, or 3 to analyze $P(u,v)$.

By Theorem 4, a degenerate parameterization represents a single point if and only if the three components of $P(u,v)$ have a common factor of degree two. However, a degenerate parameterization representing a line can occur in one of two ways: either the three components of $P(u,v)$ have exactly one common linear factor or $P(u,v)$ is unfaithful. Most of the remainder of this section is devoted to distinguishing these two cases for the three types of conic sections.

By Lüroth's Theorem (see Ref. [5]), an unfaithful parameterization $P(u,v)$ of a rational curve C can be made into a lower degree faithful parameterization $\hat{P}(s,t)$ through a reparameterization. In particular, we have the follow lemma about an unfaithful rational quadratic parameterization.

Lemma 3. *Let $P(u,v)$ be in the form of Equation (1). $P(u,v)$ is unfaithful if and only if $P(u,v)$ represents a line and the three components of $P(u,v)$ do not share a common factor. Moreover, if $P(u,v)$ is unfaithful, then $P(u,v)$ can be made faithful by a rational quadratic reparameterization.*

Proof. Suppose $P(u,v)$ is 2-to-1 almost everywhere. Then the three components of $P(u,v)$ do not share a common factor. Furthermore, there are two different pairs of parameter values $U_0 = (u_0, v_0)$ and $U_1 = (u_1, v_1)$, where $u_0 : v_0 \neq u_1 : v_1$, that give rise to the same point on $P(u,v)$ – that is, $P(u_0,v_0) = \rho P(u_1,v_1)$ for some constant ρ. Using Equation (25), we obtain

$$[V_0 - \rho V_1]M = 0,$$

where $V_0 = (u_0^2, u_0v_0, v_0^2)$ and $V_1 = (u_1^2, u_1v_1, v_1^2)$. Clearly, $V_0 - \rho V_1 \neq 0$, since $u_0 : v_0 \neq u_1 : v_1$. It follows that $\det(M) = 0$, i.e. $rank(M) < 3$. On the other hand, since the three components of $P(u,v)$ are not scalar multiples of each other, we have $rank(M) > 1$. Therefore $rank(M) = 2$. Hence by Theorem 4, $P(u,v)$ represents a line.

Conversely, suppose that $P(u,v)$ represents the line $ax + by + cw = 0$, i.e. $ap_x(u,v) + bp_y(u,v) + cp_w(u,v) = 0$, and the three components of $P(u,v)$ do not share a common factor. Without loss of generality, assume that $a \neq 0$. Then

$$p_x(u,v) = -\frac{b}{a}p_y(u,v) - \frac{c}{a}p_w(u,v). \tag{26}$$

Thus, under the reparameterization $s = s(u,v) \equiv p_y(u,v)$ and $t = t(u,v) \equiv p_w(u,v)$, $P(u,v)$ is made into the faithful parameterization

$$\hat{P}(s,t) = \left(-\frac{b}{a}s - \frac{c}{a}t, s, t\right)$$

of the same line $ax + by + cw = 0$, i.e. $\hat{P}(s(u,v), t(u,v)) = P(u,v)$. Clearly, $p_y(u,v)$ and $p_w(u,v)$ do not share a common factor; for otherwise, due to Equation(26), the three components of $P(u,v)$ would have a common factor, which is a contradiction. Thus $s = s(u,v)$ and $t = t(u,v)$ define a quadratic reparameterization, which is 2-to-1. Hence, $P(u,v)$ is unfaithful. □

With these general results in hand, we are now ready to turn our attention to degeneracies of specific types of conic sections. We begin with the following lemmas.

Lemma 4. *Suppose that* $\det(N_w) \neq 0$. *Let* γ_x *and* γ_y *be as defined in Theorems 1 and 2. Then*

$$\gamma_x = \text{Res}_{(u,v)}\{p_x(u,v), p_w(u,v)\}/\det^2(N_w),$$
$$\gamma_y = \text{Res}_{(u,v)}\{p_y(u,v), p_w(u,v)\}/\det^2(N_w).$$

Proof. Since $p_x(u,v)$, $p_y(u,v)$, $p_z(u,v)$ are quadratic polynomials,

$$\text{Res}_{(u,v)}\{p_x(u,v), p_w(u,v)\} = \begin{vmatrix} e_x & 2f_x & g_x & 0 \\ 0 & e_x & 2f_x & g_x \\ e_w & 2f_w & g_w & 0 \\ 0 & e_w & 2f_w & g_w \end{vmatrix},$$

$$\text{Res}_{(u,v)}\{p_y(u,v), p_w(u,v)\} = \begin{vmatrix} e_y & 2f_y & g_y & 0 \\ 0 & e_y & 2f_y & g_y \\ e_w & 2f_w & g_w & 0 \\ 0 & e_w & 2f_w & g_w \end{vmatrix}.$$

Now the lemma follows from straightforward verification. □

Lemma 5. *Let $P(u,v)$ be in the form of Equation (1). Suppose that* $\det(N_w) \neq 0$. *Then the three components of $P(u,v)$ share a common factor if and only if $\gamma_x = \gamma_y = \tau_{xy} = 0$, where γ_x, γ_y, and τ_{xy} are as defined in Theorems 1 and 2.*

Proof. By Lemma 4,

$$\begin{vmatrix} e_x & 2f_x & g_x & 0 \\ 0 & e_x & 2f_x & g_x \\ e_w & 2f_w & g_w & 0 \\ 0 & e_w & 2f_w & g_w \end{vmatrix} = \det^2(N_w)\gamma_x \ \text{ and } \begin{vmatrix} e_y & 2f_y & g_y & 0 \\ 0 & e_y & 2f_y & g_y \\ e_w & 2f_w & g_w & 0 \\ 0 & e_w & 2f_w & g_w \end{vmatrix} = \det^2(N_w)\gamma_y.$$

Moreover, it is also straightforward to verify that

$$\begin{vmatrix} e_x & 2f_x & g_x & 0 \\ 0 & e_y & 2f_y & g_y \\ e_w & 2f_w & g_w & 0 \\ 0 & e_w & 2f_w & g_w \end{vmatrix} + \begin{vmatrix} e_y & 2f_y & g_y & 0 \\ 0 & e_x & 2f_x & g_x \\ e_w & 2f_w & g_w & 0 \\ 0 & e_w & 2f_w & g_w \end{vmatrix} = 2\det^2(N_w)\tau_{xy}. \tag{27}$$

Thus

$$\begin{aligned} &\operatorname{Res}_{(u,v)}\{p_x(u,v) + p_y(u,v),\ p_w(u,v)\} \\ &= \begin{vmatrix} e_x+e_y & 2f_x+2f_y & g_x+g_y & 0 \\ 0 & e_x+e_y & 2f_x+2f_y & g_x+g_y \\ e_w & 2f_w & g_w & 0 \\ 0 & e_w & 2f_w & g_w \end{vmatrix} \\ &= \begin{vmatrix} e_x & 2f_x & g_x & 0 \\ 0 & e_x & 2f_x & g_x \\ e_w & 2f_w & g_w & 0 \\ 0 & e_w & 2f_w & g_w \end{vmatrix} + \begin{vmatrix} e_y & 2f_y & g_y & 0 \\ 0 & e_y & 2f_y & g_y \\ e_w & 2f_w & g_w & 0 \\ 0 & e_w & 2f_w & g_w \end{vmatrix} \\ &+ \begin{vmatrix} e_x & 2f_x & g_x & 0 \\ 0 & e_y & 2f_y & g_y \\ e_w & 2f_w & g_w & 0 \\ 0 & e_w & 2f_w & g_w \end{vmatrix} + \begin{vmatrix} e_y & 2f_y & g_x & 0 \\ 0 & e_x & 2f_x & g_x \\ e_w & 2f_w & g_w & 0 \\ 0 & e_w & 2f_w & g_w \end{vmatrix} \\ &= \det^2(N_w)(\gamma_x + \gamma_y + 2\tau_{xy}). \end{aligned} \tag{28}$$

Now suppose that $\gamma_x = \gamma_y = \tau_{xy} = 0$. Since, by Lemma 4,

$$\operatorname{Res}_{(u,v)}\{p_x(u,v), p_w(u,v)\} = \det^2(N_w)\gamma_x = 0,$$

$p_x(u,v)$ and $p_w(u,v)$ share a common factor $q_1(u,v)$. Similarly, $p_y(u,v)$ and $p_w(u,v)$ share a common factor $q_2(u,v)$. If either $q_1(u,v)$ or $q_2(u,v)$ is quadratic, then obviously $p_x(u,v)$, $p_y(u,v)$, and $p_w(u,v)$ share a common factor. If both $q_1(u,v)$ and $q_2(u,v)$ are linear and equal up to a constant multiple, then again $p_x(u,v)$, $p_y(u,v)$, and $p_w(u,v)$ share a common factor. Now suppose that $q_1(u,v)$ and $q_2(u,v)$ are linear and not scalar multiples of each other. In this case $q_1(u,v)$ and $q_2(u,v)$ must be the two linear factors of $p_w(u,v)$.

Moreover, since $\gamma_x = \gamma_y = \tau_{xy} = 0$, by Equation (28), $p_x(u,v) + p_y(u,v)$ and $p_w(u,v)$ have at least one common linear factor, which, without loss of generality, can be assumed to be $q_1(u,v)$. But then $q_1(u,v)$ is also a factor of $p_y(u,v)$, since $p_y(u,v) = (p_x(u,v) + p_y(u,v)) - p_x(u,v)$. Hence, $q_1(u,v)$ is a common factor of $p_x(u,v)$, $p_y(u,v)$, and $p_w(u,v)$.

Conversely, suppose $p_x(u,v)$, $p_y(u,v)$, and $p_w(u,v)$ share a common factor. Then

$$\begin{aligned}
0 &= \mathrm{Res}_{(u,v)}\{p_x(u,v), p_w(u,v)\} = \det{}^2(N_w)\gamma_x,\\
0 &= \mathrm{Res}_{(u,v)}\{p_y(u,v), p_w(u,v)\} = \det{}^2(N_w)\gamma_y,\\
0 &= \mathrm{Res}_{(u,v)}\{p_x(u,v) + p_y(u,v),\ p_w(u,v)\} = \det{}^2(N_w)(\gamma_x + \gamma_y + 2\tau_{xy}).
\end{aligned}$$

Hence, $\gamma_x = \gamma_y = \tau_{xy} = 0$, since $\det(N_w) \neq 0$. □

Lemma 6. *Let $P(u,v)$ be in the form of Equation (1). If $\det(N_w) > 0$, then $p_w(u,v)$ has no real linear factor.*

Proof. This result is obvious, since $\det(N_w) = -\Delta/4$, where $\Delta = 4f_w^2 - 4e_w g_w$ is the discriminant of $p_w(u,v)$. □

Theorem 5. *(The elliptic case - I) Let $P(u,v)$ be in the form of Equation (1). Suppose that $\det(N_w) > 0$. Then the following conditions are equivalent:*

a. *$P(u,v)$ represents a point.*
b. $rank(M) = 1$.
c. $\gamma_x = \gamma_y = 0$.

Proof. The equivalence of (a) and (b) follows from Theorem 4. Therefore, we just need to prove the equivalence of (b) and (c).

Suppose that $rank(M) = 1$. Then the three components of $P(u,v)$ are scalar multiples of each other. It follows, by Lemma 4, that $\gamma_x = \gamma_y = 0$.

Conversely, suppose $\gamma_x = \gamma_y = 0$. Then, by Lemma 4, $p_x(u,v)$ and $p_w(u,v)$ share a common factor, and $p_y(u,v)$ and $p_w(u,v)$ share a common factor. Therefore, by Lemma 6, $p_x(u,v)$ and $p_w(u,v)$ are scalar multiples of each other, and $p_y(u,v)$ and $p_w(u,v)$ are scalar multiples of each other. Thus the three components of $P(u,v)$ are scalar multiples of each other. Hence, $rank(M) = 1$. □

Theorem 6. *(The elliptic case - II) Let $P(u,v)$ be in the form of Equation (1). Suppose that $\det(N_w) > 0$. Then the following conditions are equivalent:*

a. *$P(u,v)$ represents a line.*
b. $rank(M) = 2$.
c. $\det(M) = 0$ *and* $\gamma_x^2 + \gamma_y^2 \neq 0$.

d. $P(u,v)$ is not a faithful parameterization.

Proof.

$(a) \rightarrow (b)$: Follows by Theorem 4.

$(b) \rightarrow (c)$: First, we have $\det(M) = 0$, since $rank(M) = 2$. Moreover, since $rank(M) \neq 1$, the three components of $P(u,v)$ are not scalar multiples of each other. Without loss of generality, assume that $p_x(u,v)$ and $p_w(u,v)$ are not scalar multiples of each other. It follows by Lemma 6 that $p_x(u,v)$ and $p_w(u,v)$ do not have a common factor. Therefore, by Lemma 4, $\gamma_x \neq 0$. Hence, $\gamma_x^2 + \gamma_y^2 \neq 0$.

$(c) \rightarrow (d)$: Since $\gamma_x^2 + \gamma_y^2 \neq 0$, by Lemma 4, the three components of $P(u,v)$ do not share a common factor. Thus, by Theorem 5, we have $rank(M) > 1$. On the other hand, since $\det(M) = 0$, we have $rank(M) < 3$. Therefore $rank(M) = 2$, so, by Theorem 4, $P(u,v)$ represents a line. Hence, by Lemma 3, $P(u,v)$ is unfaithful.

$(d) \rightarrow (a)$: Follows by Lemma 3. □

Note that in the elliptic case (i.e. $\det(N_w) > 0$), the three components of $P(u,v)$ cannot share a common real linear factor. That is, in the elliptic case, this degenerate case does not occur.

If either γ_x or γ_y but not both are zero, then $P(u,v)$ represents a line since, by Lemma 6, either $p_x(u,v)$ or $p_y(u,v)$ is a scalar multiple of $p_w(u,v)$. In fact, in this case the line is in a special position: either vertical (when $\gamma_x = 0$) or horizontal (when $\gamma_y = 0$). Note too that in this case, since the three components of $P(u,v)$ do not have a common factor, by Lemma 3, $P(u,v)$ is necessarily unfaithful.

Theorem 7. *(The hyperbolic case - I) Let $P(u,v)$ be in the form of Equation (1). Suppose that $\det(N_w) < 0$, i.e. the hyperbolic case. Let $\gamma_x, \gamma_y, \tau_{xy}$ and δ_{xy} be defined as in Theorem 2. Then the following conditions are equivalent:*

a. The three components of $P(u,v)$ share a common factor.
b. $\gamma_x = \gamma_y = \det(M) = 0$.
c. $\gamma_x = \gamma_y = \tau_{xy} = 0$.
d. $\delta_{xy} = 0$.

In this case $P(u,v)$ represents a line if $rank(M) = 2$, and $P(u,v)$ represents a single point if $rank(M) = 1$.

Proof. The equivalence of (a) and (c) is proved in Lemma 5. We next show the equivalence of (a) and (b), and then the equivalence of (c) and (d).

$(a) \rightarrow (b)$: This result follows from Lemma 4 and Theorem 4.

$(b) \rightarrow (a)$: Since $\gamma_x = \gamma_y = 0$, it follows by Lemma 4 that $p_x(u,v)$ and $p_w(u,v)$ share a common factor, and $p_y(u,v)$ and $p_w(u,v)$ share a common factor. Moreover, since $\det(M) = 0$, the three components of $P(u,v)$ are linearly dependent – that is,

$$ap_x(u,v) + bp_y(u,v) + cp_w(u,v) = 0,$$

where a, b, c are not all zero. Furthermore, it is evident that a and b are not both zero; for otherwise, since $c \neq 0$, we would have $p_w(u,v) = 0$, which is a contradiction. Thus, without loss of generality, we can assume that $a \neq 0$. Then from

$$p_x(u,v) = -\frac{b}{a}p_y(u,v) - \frac{c}{a}p_w(u,v),$$

it follows that the common factor of $p_y(u,v)$ and $p_w(u,v)$ is also a factor of $p_x(u,v)$. Hence, the three components of $P(u,v)$ share a common factor.

$(c) \to (d)$: This result follows because by definition (see Theorem 2)

$$\delta_{xy} = \sqrt{(\gamma_x - \gamma_y)^2 + 4\tau_{xy}^2}. \tag{29}$$

$(d) \to (c)$: Suppose that $\delta_{xy} = 0$. Then from Equation (29), we obtain $\gamma_x - \gamma_y = 0$ and $\tau_{xy} = 0$. Moreover, when $\delta_{xy} = 0$, we have $\gamma_x + \gamma_y = 2\ell_1^2 \geq 0$ from (14), and $\gamma_x + \gamma_y = -2\ell_2^2 \leq 0$ from (15). Thus $\gamma_x + \gamma_y = 0$. It follows that $\gamma_x = \gamma_y = \tau_{xy} = 0$.

Finally, the last statement of the theorem follows from Theorem 4. □

Theorem 8. *(The hyperbolic case - II) Let $P(u,v)$ be in the form of Equation (1). Suppose that* $\det(N_w) < 0$. *Then the following conditions are equivalent:*

a. *$P(u,v)$ is not a faithful parameterization.*
b. $\det(M) = 0$ *and* $\gamma_x^2 + \gamma_y^2 \neq 0$.
c. $\det(M) = 0$ *and* $\delta_{xy} \neq 0$.

Proof. We are going to show separately the equivalence of (a) and (b) and the equivalence of (a) and (c).

$(a) \to (b)$: Suppose that $P(u,v)$ is not a faithful parameterization. Then by Theorem 4 we have $\det(M) = 0$. If $\gamma_x^2 + \gamma_y^2 = 0$, i.e. $\gamma_x = \gamma_y = 0$, then by Theorem 7 the three components of $P(u,v)$ have a common factor; however, this is a contradiction, since by Lemma 3, in an unfaithful parameterization, the three components of $P(u,v)$ do not have a common factor. Hence, $\gamma_x^2 + \gamma_y^2 \neq 0$.

$(b) \to (a)$: Since $\det(M) = 0$, by Theorem 4, $P(u,v)$ represents either a single point or a line. Since $\gamma_x^2 + \gamma_y^2 \neq 0$, by Lemma 5, the three components of $P(u,v)$ do not have a common factor. Therefore, $P(u,v)$ represents a line, not a single point, and by Lemma 3, $P(u,v)$ is unfaithful.

$(a) \to (c)$: Suppose that $P(u,v)$ is not a faithful parameterization. First, by Theorem 4, we have $\det(M) = 0$. Furthermore, by Lemma 3, the three components of $P(u,v)$ do not have a common factor. But if $\delta_{xy} = 0$, then by Theorem 7 the three components of $P(u,v)$ share a common factor. This is a contradiction. Hence, $\delta_{xy} \neq 0$.

$(c) \to (a)$: Suppose that $\det(M) = 0$ and $\delta_{xy} \neq 0$. Since $\det(M) = 0$, by Theorem 4, $P(u,v)$ is either unfaithful or its three components share at least one common factor. But in the latter case, by Theorem 7, $\delta_{xy} = 0$. This is a contradiction. Hence, $P(u,v)$ is unfaithful. □

Remark: Note that $\det(M) = 0$ and $\tau_{xy} \neq 0$ is a sufficient condition for $P(u,v)$ to be unfaithful, but not a necessary one. To see the sufficiency, we suppose that $\det(M) = 0$ and $\tau_{xy} \neq 0$. Then, by Theorem 4, $P(u,v)$ is degenerate. Moreover, it is evident that $P(u,v)$ is unfaithful; otherwise, the three components of $P(u,v)$ would share a common factor, which by Lemma 5 contradicts the fact that $\tau_{xy} \neq 0$. To see that this condition is not a necessary one for $P(u,v)$ to be unfaithful, consider the parameterization $\tilde{P}(u,v) = (u^2 - v^2, v^2, u^2 - v^2)$. Clearly, $\det(N_w) < 0$. Since $\tilde{P}(u,v)$ represents the line $x - w = 0$ and the three components of $P(u,v)$ share no common factor, by Lemma 3, $P(u,v)$ is unfaithful. However, using Equation (27), it is easy to verify that $\det(N_w)\tau_{xy} = 0$. Hence, $\tau_{xy} = 0$, since $\det(N_w) < 0$.

For the parabolic case, i.e. when $\det(N_w) = 0$, define

$$L^2 = \mathrm{tr}^2(N_x N_w^*) + \mathrm{tr}^2(N_y N_w^*).$$

Then we have the following results.

Lemma 7. *Let $P(u,v)$ be in the form of Equation (1). Suppose that* $\det(N_w) = 0$. *Then the three components of $P(u,v)$ share a common factor if and only if $L^2 = 0$.*

Proof. Since $\det(N_w) = 0$,

$$P_w(u,v) \equiv e_w u^2 + 2f_w uv + g_w v^2 = w_0(v_0 u - u_0 v)^2$$

for some real numbers $w_0 \neq 0$, and v_0, u_0 with $u_0^2 + v_0^2 \neq 0$. Thus

$$e_w = w_0 v_0^2, \quad f_w = -w_0 u_0 v_0, \quad g_w = w_0 u_0^2.$$

The resultant of $p_x(u,v) = e_x u^2 + 2f_x uv + g_x v^2$ and $\ell_w(u,v) \equiv v_0 u - u_0 v$ is

$$\begin{aligned}\mathrm{Res}_{(u,v)}\{p_x(u,v), \ell_w(u,v)\} &= \begin{vmatrix} e_x & 2f_x & g_x \\ v_0 & -u_0 & 0 \\ 0 & v_0 & -u_0 \end{vmatrix} \\ &= e_x u_0^2 + g_x v_0^2 + 2f_x u_0 v_0 = \frac{1}{w_0}(e_x g_w + e_w g_x - 2f_x f_w) \\ &= \frac{1}{w_0}\mathrm{tr}(N_x N_w^*).\end{aligned}$$

Similarly,

$$\mathrm{Res}_{(u,v)}\{p_y(u,v), \ell_w(u,v)\} = \frac{1}{w_0}\mathrm{tr}(N_y N_w^*).$$

Thus $p_x(u,v)$ and $p_y(u,v)$ share the common factor $v_0 u - u_0 v$ with $p_w(u,v)$ if and only if $\mathrm{tr}(N_x N_w^*) = 0$ and $\mathrm{tr}(N_y N_w^*) = 0$, i.e. if and only if $L^2 = 0$. □

Theorem 9. *(The parabolic case - I) Let $P(u,v)$ be in the form of Equation (1). Suppose that* $\det(N_w) = 0$. *Then the following conditions are equivalent:*

a. The three components of $P(u,v)$ are scalar multiples of each other, i.e. $P(u,v)$ represents a single point.

b. $rank(M) = 1$.
c. $L^2 = \det(N_x) = \det(N_y) = 0$.

Proof.
$(a) \rightarrow (b)$: This result follows from Theorem 4.

$(b) \rightarrow (c)$: Since $rank(M) = 1$, the three components of $P(u,v)$ are scalar multiples of each other; therefore, both $p_x(u,v)$ and $p_y(u,v)$ are perfect squares, since $p_w(u,v)$ is a perfect square. Thus $\det(N_x) = \det(N_y) = 0$. Furthermore, it follows from Lemma 7 that $L^2 = 0$.

$(c) \rightarrow (a)$: Since $\det(N_x) = \det(N_y) = 0$, both $p_x(u,v)$ and $p_y(u,v)$ are perfect squares. Since $L^2 = 0$, by Lemma 7, the three components of $P(u,v)$ share a common factor. It follows that the three components of $P(u,v)$ are scalar multiples of each other. Hence, $P(u,v)$ represents a point. □

Theorem 10. *(The parabolic case - II) Let $P(u,v)$ be in the form of Equation (1). Suppose that* $\det(N_w) = 0$. *Then the following conditions are equivalent:*

a. The three components of $P(u,v)$ share exactly one linear factor.
b. $L^2 = 0$ *and* $\det^2(N_x) + \det^2(N_y) \neq 0$.
c. $L^2 = 0$ *and* $rank(M) = 2$.

Proof.
$(a) \rightarrow (b)$: Suppose that the three components of $P(u,v)$ share exactly one linear factor. Then $L^2 = 0$, by Lemma 7. Since $P(u,v)$ does not represent a single point, by Theorem 9, $\det^2(N_x) + \det^2(N_y) \neq 0$.

$(b) \rightarrow (c)$: Since $L^2 = 0$, by Lemma 7, $P(u,v)$ is degenerate. Thus, by Theorem 4, $\det(M) = 0$, i.e. $rank(M) < 3$. On the other hand, since $\det^2(N_x) + \det^2(N_y) \neq 0$, by Theorem 9, $rank(M) \neq 1$. Since $rank(M) > 0$ by assumption, it follows that $rank(M) = 2$.

$(c) \rightarrow (a)$: Since $rank(M) = 2$, by Theorem 4, $P(u,v)$ represents a line. Since $L^2 = 0$, by Lemma 7, the three components of $P(u,v)$ share a common factor; but, by Theorem 9, this common factor cannot be quadratic, since $rank(M) \neq 1$. Hence, the three components of $P(u,v)$ share exactly one linear factor. □

Theorem 11. *(The parabolic case - III) Let $P(u,v)$ be in the form of Equation (1). Suppose that* $\det(N_w) = 0$. *Then the following conditions are equivalent:*

a. $P(u,v)$ is not a faithful parameterization.
b. $\det(M) = 0$ *and* $L^2 \neq 0$.

Proof.
$(a) \rightarrow (b)$: Suppose that $P(u,v)$ is unfaithful. Then by Lemma 3 the three components of $P(u,v)$ do not have a common factor. Thus, by Lemma 7, $L^2 \neq 0$. Moreover, by Theorem 4, $\det(M) = 0$.

$(b) \rightarrow (a)$: Since $\det(M) = 0$, by Theorem 4, $P(u, v)$ is degenerate. However, since $L^2 \neq 0$, by Lemma 7, the three components of $P(u, v)$ do not have a common factor. It follows, by Theorem 4 again, that $P(u, v) = 0$ represents a line, and, by Lemma 3, that $P(u, v)$ is not a faithful parameterization. □

7. Rational Quadratic Bézier Curves

In some applications a conic section may appear in rational quadratic Bézier form

$$\hat{P}(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2,$$

with homogeneous control points $P_i = (x_i, y_i, w_i)$, $i = 0, 1, 2$. Through the reparameterization $s = (1-t)/t$, the same curve can be rewritten in the form

$$\bar{P}(s) = s^2 P_0 + 2sP_1 + P_2.$$

Thus one just needs to put $E = P_0$, $F = P_1$, and $G = P_2$, in order to apply the formulas in the results in the preceding sections, without having to convert from the Bézier form to the power form.

A conic section in rational Bézier form is completely characterized by its affine control points Q_j and its scalar weights w_j, $j = 0, 1, 2$. Thus the geometric characteristics of such conics are functions of these six parameters. For example, it follows easily from Theorems 1, 2 and discussion in the previous paragraph that the center C of an ellipse or hyperbola is given by

$$C = \frac{w_0 w_2 Q_2 - 2w_1^2 Q_1 + w_0 w_2 Q_0}{2(w_0 w_2 - w_1^2)}.$$

Similarly the unit direction vector D of the symmetry axis of a parabola is parallel to the vector

$$D^* = w_0 w_2 Q_2 - 2w_1^2 Q_1 + w_0 w_2 Q_0.$$

But for a parabola, $\det(N_w) = w_0 w_2 - w_1^2 = 0$, so $w_0 w_2 = w_1^2$. Hence, D is actually parallel to $Q_2 - 2Q_1 + Q_0$.

Simple closed formulas for other geometric features of conic sections in rational Bézier form are not so easy to derive since, for example, the formulas in Theorems 1, 2 for the semi-major and semi-minor axis vectors are in terms of polar, rather than rectangular, coordinates. Finding simple closed formulas for these geometric characteristics in terms of Bézier control points and weights is still an open problem.

8. Conclusions

We have presented an algebraic approach based on invariants to computing the geometric characteristics of conic sections directly from their rational quadratic parameterizations. In contrast with existing algorithmic solutions, our treatment is elementary and yields simple explicit formulas for all the geometric characteristics of any conic in terms of these invariants. The algebraic derivations of these invariant characterizations take several pages, but the formulas themselves are concise, simple to summarize (Theorems 1-3), and easy to program. These algebraic invariants also have clear geometric meanings, and are used here as well to give a complete characterization of different degenerate rational quadratic parameterizations of conic sections, which is another contribution of this paper.

When using our results to compute the geometric characteristics of a conic section from one of its quadratic parameterizations $P(u,v)$, one first needs to check $\det(M)$ (Theorem 4). If $\det(M) \neq 0$, the conic is proper. Then the type of the conic can be determined by testing the sign of $\det(N_w)$ (Lemma 2). For the three cases where the conic is an ellipse, a hyperbola, or a parabola, its geometric characteristics can be computed using the closed formulas provided in Theorem 1, Theorem 2, or Theorem 3. If $\det(M) = 0$, the conic is degenerate, and reduces to a line or a point. In this case, one may further distinguish between the different degenerate cases by applying the results in Section 6.

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